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# Positive Solutions of Quasilinear Elliptic Equations in Exterior Domains

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## 1. INTRODUCTION

Our main purpose is to prove the existence of a positive solution of the quasilinear second order elliptic differential equation (1) below in an exterior domain in Euclidean space $R^n$, $n \geqslant 2$. More explicitly, if (1) has a positive subsolution $w$ and a positive supersolution $v$ such that $w(x) \leqslant v(x)$ throughout an exterior domain $G_a$ (notation in Section 2), Theorem 3.3 establishes the existence of a solution $u$ of (1) in $G_a$ such that $w(x) \leqslant u(x) \leqslant v(x)$ for all $x \in G_a$. An analogous theorem for bounded domains due to Nagumo [7] is employed in Lemma 3.1 to construct a sequence of positive solutions of (1) in annular domains $G(a, a+j)$, $j = 1, 2,\ldots$. We then prove in Lemma 3.2 and Theorem 3.3 that this sequence converges to a positive solution of (1) in $G_a$ by means of interior Schauder estimates, $L^p$-space estimates, and a priori interior estimates on the gradient of a solution of (1) in a bounded domain.

Theorem 3.3 is applied in Section 4 to give sufficient conditions for a quasilinear Schrödinger equation (8) to have a solution in an exterior domain in $R^n$ satisfying $0 \leqslant u(x) \leqslant |x|^q$, $q = 2 - n + \epsilon$, for arbitrary $\epsilon$ in $(0, n-2)$, $n \geqslant 3$; and a similar result is given for $n = 2$. Section 5 indicates the procedure for reduction of nonoscillation problems for (8) to corresponding problems for quasilinear *ordinary* differential equations or inequalities. In particular, criteria are given for the existence of a bounded positive solution of the generalized Emden–Fowler equation in some exterior domain. A necessary and sufficient

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condition for nonoscillation of a sublinear Emden–Fowler equation in dimensions $n \geqslant 3$ is deduced from a recent result of Kitamura and Kusano [5], and an analogous superlinear result from the authors' earlier result in [9].

## 2. Preliminaries

Let $|x|$ denote the Euclidean norm of a point $x = (x_1, \ldots, x_n)$ in real $n$-dimensional Euclidean space $R^n$. Define

$$S_r = \{x \in R^n : |x| = r\},$$

$$G_r = \{x \in R^n : |x| > r\},$$

$$G(r, s) = \{x \in R^n : r < |x| < s\}, \qquad 0 < r < s.$$

The Hölder norms of a function $u\colon \bar{M} \to R^1$ on the closure $\bar{M}$ of a bounded domain $M \subset R^n$ are defined by

$$\| u \|_{\alpha, \bar{M}} = \sup_{\substack{x, y \in \bar{M} \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}$$

$$\| u \|_{m+\alpha, \bar{M}} = \sum_{|i|=m} \| D^i u \|_{\alpha, \bar{M}} + \sup_{x \in \bar{M}} \sum_{|i| \leqslant m} | D^i u(x) | ,$$

$$0 < \alpha < 1, \qquad m = 1, 2, \ldots,$$

where $i$ denotes a multi-index of length $|i|$. For convenience $\| u \|_{0, \bar{M}}$ is defined as $\sup_{x \in \bar{M}} |u(x)|$. As usual, $C^{m+\alpha}(\bar{M})$ denotes the space of all functions $u\colon \bar{M} \to R^1$ such that $\| u \|_{m+\alpha, \bar{M}}$ is finite. The boundary $\partial M$ of $M$ is said to belong to class $C^{m+\alpha}$ whenever every $x \in \partial M$ has a neighborhood $N$ such that $\partial M \cap N$ can be represented in the form

$$x_i = h(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

for some integer $i$, where $h \in C^{m+\alpha}$ on the closure of an appropriate bounded domain.

The quasilinear elliptic differential equation

$$\begin{gathered} Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x)\, D_i D_j u = f(x, u, \nabla u), \qquad x \in \Omega \\ D_i = \partial / \partial x_i, \qquad \nabla = (D_1, \ldots, D_n), \end{gathered} \tag{1}$$

is under consideration in an exterior domain $\Omega \subset R^n$, i.e. $G_a \subset \Omega$ for some positive number $a$.

ASSUMPTIONS

(i) Each $a_{ij} \in C^{2+\alpha}(\overline{M})$ and the matrix $(a_{ij}(x))$ is uniformly positive definite in every bounded domain $M \subset \Omega$ (uniform ellipticity condition);

(ii) There is a number $\alpha$ in $(0, 1)$, fixed in the sequel, such that $f \in C^{\alpha}(\overline{M} \times \bar{J} \times \overline{N})$ for every bounded domain $M \subset \Omega$, every bounded interval $J \subset R^1$, and every bounded domain $N \subset R^n$;

(iii) For every bounded subdomain $M$ of $\Omega$, there exists a nonnegative continuous function $g_M$ such that

$$|f(x, u, p)| \leqslant g_M(|u|)(1 + |p|^2), \qquad x \in \overline{M}, \quad u \in R^1, \quad p \in R^n$$

(Nagumo condition).

A *solution* $u$ of (1) in $\Omega$ is defined to be a function $u \in C^{2+\alpha}(\overline{M})$ for every bounded subdomain $M \subset \Omega$, with $\alpha$ as in (1), such that $u$ satisfies (1) at every point $x \in \Omega$. Subsolutions of (1), i.e. functions $u$ satisfying $Lu \geqslant f(x, u, \nabla u)$, and supersolutions are defined similarly.

The theorem below will be needed in the next section; a proof is given by Ladyzhenskaya and Ural'tseva [6, p. 369].

THEOREM 0. *For a bounded domain $S \subset \Omega$, let $u \in C^2(\bar{S})$ be a solution of* (1) *in $S$ under the above assumptions. Suppose that $R$ is a bounded domain with $\bar{R} \subset S$. Then there exists a constant $K$, independent of $u$, such that*

$$\max_{x \in \bar{R}} |\nabla u(x)| \leqslant K \max_{x \in \bar{S}} |u(x)|,$$

*where $K$ depends only on $n$, $g_M$,* dist$(R, S)$, *and the Hölder constants for the coefficients $a_{ij}$ in* (1).

## 3. EXISTENCE OF A POSITIVE SOLUTION

If (1) has a positive subsolution $w$ and a positive supersolution $v$ such that $w(x) \leqslant v(x)$ for all $x$ in an exterior domain $G_a$, the main Theorem 3.3 below establishes the existence of a positive solution of (1) in $G_a$ that is squeezed between $w(x)$ and $v(x)$. Theorem 0 and two preliminary lemmas below are required in the proof.

LEMMA 3.1. *Let $\Omega$, $L$, and $\alpha$ be as in* (1) *and suppose that a positive number $a$ is fixed such that $G_a \subset \Omega$. If there exist positive solutions $v$ and $w$ of $Lv \leqslant f(x, v, \nabla v)$ and $Lw \geqslant f(x, w, \nabla w)$, respectively, in $G_a$ such that $w(x) \leqslant v(x)$ for all $x \in G_a \cup S_a$, then there exists a sequence of functions $u_i$ in $G_a \cup S_a$ with the following properties:*

(A) $u_j \in C^{2+\alpha}(\overline{G(a, a+j)})$;

(B) $u_j(x) = v(x)$ *if* $x \in S_a \cup S_{a+j} \cup G_{a+j}$;

(C) $Lu_j = f(x, u_j, \nabla u_j)$ *in* $G(a, a+j)$; *and*

(D) $w(x) \leqslant u_j(x) \leqslant v(x)$ *if* $x \in G_a \cup S_a$, $j = 1, 2, \ldots$.

*Proof.* Since $w$ is a subsolution and $v$ is a supersolution of (1) with $w(x) \leqslant v(x)$ in $\overline{G(a, a+j)}$, a theorem of Nagumo [7] (see also Schmitt [11, Theorem 3.2]) guarantees the existence of a solution $u = U_j \in C^{2+\alpha}(\overline{G(a, a+j)})$ of the boundary problem

$$\begin{aligned} Lu &= f(x, u, \nabla u) \qquad &&\text{in} \qquad G(a, a+j) \\ u(x) &= v(x) &&\text{on} \qquad S_a \cup S_{a+j} \end{aligned}$$

satisfying $w(x) \leqslant U_j(x) \leqslant v(x)$ for all $x \in \overline{G(a, a+j)}$. The extension $u_j(x)$ of $U_j(x)$ to $G_a \cup S_a$ defined by $u_j(x) = v(x)$ for $|x| > a + j$ then has all the required properties (A), (B), (C), and (D) of Lemma 3.1.

LEMMA 3.2. *Let* $\{u_j\}$ *be the sequence in Lemma* 3.1. *Let* $i$ *be a positive integer and* $M$ *be an arbitrary bounded domain such that* $\overline{M} \subset G(a, a+i)$. *Then there exists a positive constant* $K_0$, *depending on* $\alpha$, $n$, $i$, $v$, *and* $w$ *but independent of* $j$, *such that*

$$\| u_j \|_{2+\alpha, \overline{M}} \leqslant K_0 \qquad \text{for all } j \geqslant i. \tag{2}$$

*Proof.* Let $Q$ and $R$ be bounded domains such that $\overline{M} \subset Q$, $\overline{Q} \subset R$, $\overline{R} \subset G(a, a+i)$, $\partial Q \in C^{2+\alpha}$, and $\partial R \in C^{2+\alpha}$. For $j \geqslant i$, property (C) of Lemma 3.1 shows that $u_j$ satisfies $Lu_j = f(x, u_j, \nabla u_j)$ in $G(a, a+i)$. By Theorem 0, there exists a constant $K$ independent of $u_j$ such that

$$\max_{x \in \overline{R}} | \nabla u_j(x)| \leqslant K \max_{x \in \overline{G(a, a+i)}} | u_j(x)| .$$

Property (D) of Lemma 3.1 then implies that $\{\nabla u_j\}$ as well as $\{u_j\}$ is uniformly bounded on $\overline{R}$.

Let $v_j(x)$, $j \geqslant i$, be the unique solution of the *linear* boundary problem

$$\begin{aligned} Lv &= f(x, u_j(x), \nabla u_j(x)), \qquad && x \in R \\ v(x) &= 0, && x \in \partial R. \end{aligned} \tag{3}$$

Define $f_j(x) = f(x, u_j(x), \nabla u_j(x))$, which is a uniformly bounded sequence on $\overline{R}$ on account of the uniform boundedness of $\{u_j\}$ and $\{\nabla u_j\}$. For any number $p > 1$ it follows that there exists a positive constant $K_1$, independent of $j$, such that $\| f_j \|_{L^p(R)} \leqslant K_1$ for all $j \geqslant i$. The norm of the solution $v_j(x)$ of (3) in the Sobolev space $W_p{}^2(R)$ therefore satisfies

$$\| v_j \|_{W_p{}^2(R)} \leqslant K_2$$

for some constant $K_2$ independent of $j$ by $L^p$ estimates of Agmon, Douglis, and Nirenberg [1, p. 704]. With the choice $p = n/(1 - \alpha)$, the Sobolev embedding lemma [6, p. 43] shows that $v_j \in C^{1+\alpha}(\bar{R})$ and

$$\| v_j \|_{1+\alpha,\bar{R}} \leqslant K_3 \| v_j \|_{W_p{}^2(R)} \leqslant K_3 K_2 , \qquad j \geqslant i \tag{4}$$

for another positive constant independent of $j$.

Let $w_j(x)$ be the unique solution of the boundary problem

$$\begin{aligned} Lw(x) &= 0 & x &\in R \\ w(x) &= u_j(x), & x &\in \partial R, \quad j \geqslant i. \end{aligned} \tag{5}$$

Since $u_j(x) \leqslant v(x)$ for all $x \in \bar{R}$ and for all $j \geqslant i$ by Lemma 3.1, the maximum principle for elliptic equations [10] shows that there exists a constant $K_4$ independent of $j$ so that $\| w_j \|_{0,\bar{R}} \leqslant K_4$ for all $j \geqslant i$. Classical interior Schauder estimates for (5), see [4, p. 335] for example, then yield

$$\| w_j \|_{2+\alpha,\bar{Q}} \leqslant K_5 \| w_j \|_{0,\bar{R}} \leqslant K_5 K_4 , \tag{6}$$

where the constant $K_5$ is again independent of $j$.

It follows from (3) and (5) that $u = v_j + w_j$ is a solution of the *linear* boundary problem

$$\begin{aligned} Lu &= f(x, u_j(x), \nabla u_j(x)), & x &\in R \\ u(x) &= u_j(x), & x &\in \partial R. \end{aligned} \tag{7}$$

However, $u = u_j$ also is a solution of (7) by Lemma 3.1, and hence $u_j = v_j + w_j$ in $\bar{R}$ by the standard uniqueness theorem for (7). It is then a consequence of (4) and (6) that $\| u_j \|_{1+\alpha,\bar{Q}} \leqslant K_6$ for $j \geqslant i$. In view of the regularity assumption (ii) the function $f_j(x) = f(x, u_j(x), \nabla u_j(x))$ in (7) satisfies $\| f_j \|_{\alpha,\bar{Q}} \leqslant K_7$ for all $j \geqslant i$, where $K_7$ is again independent of $j$. Since $u_j(x)$ is a solution of $Lu(x) = f_j(x)$ for $x \in \bar{Q}$ by (7), the interior Schauder estimate [4, p. 335]

$$\| u_j \|_{2+\alpha,\bar{M}} \leqslant K_8 [\| u_j \|_{0,\bar{Q}} + \| f_j \|_{\alpha,\bar{Q}}]$$

implies the conclusion (2) of Lemma 3.2.

THEOREM 3.3. *Under the hypotheses of Lemma* 3.1, (1) *has a solution $u$ in $G_a$ such that $w(x) \leqslant u(x) \leqslant v(x)$ throughout $G_a \cup S_a$.*

*Proof.* Let $\{u_j\}$ be the sequence in Lemma 3.1. For each integer $i = 1, 2, \ldots$ define

$$M_i = G\left(a + \frac{1}{3i}, a + i - \frac{1}{3i}\right).$$

Then $\overline{M}_i \subset G(a, a+i)$ and by Lemma 3.2 there exists a positive constant $K_0$, independent of $j$, such that $\| u_j \|_{2+\alpha, \overline{M}_i} \leqslant K_0$ for all $j \geqslant i$. The compactness of the injection $C^{2+\alpha}(\overline{M}_1) \to C^2(\overline{M}_1)$ then implies that $\{u_j : j \geqslant 1\}$ has a subsequence $\{u_j^1\}$ which converges uniformly in the $C^2(\overline{M}_1)$ norm to a function $u^1$ on $\overline{M}_1$. Define $u_j^0 = u_j$ for convenience and define $\{u_j^i\}$ inductively to be a subsequence of $\{u_j^{i-1}\}$ which converges uniformly in the $C^2(\overline{M}_i)$ norm to a function $u^i$ on $\overline{M}_i$, $i = 1, 2, \ldots$. Define $u$ in $G_a$ by $u(x) = u^i(x)$ if $x \in \overline{M}_i$; this definition is consistent since $M_{i+1} \supset M_i$ and $u^{i+1} = u^i$ on $\overline{M}_i$ obviously for each $i = 1, 2, \ldots$.

We shall show that the required solution $u$ of (1) in $G_a$ is given by

$$u(x) = u^i(x) \quad \text{if} \quad x \in M_i, \qquad i = 1, 2, \ldots.$$

First note that the diagonal sequence $\{u_j^j(x)\}$ converges to $u(x)$ for all $x \in G_a$. For any bounded domain $\overline{M} \subset G_a$, $\overline{M} \subset \overline{M}_i$ for some integer $i$, and hence $\{u_j^j : j \geqslant i\}$ (being a subsequence of $\{u_j^i : j \geqslant i\}$), converges uniformly in the $C^2(\overline{M})$ norm to $u^i = u$ on $\overline{M}$. In particular $u_j^j$ and $Lu_j^j$ converge uniformly on $\overline{M}$ to $u$ and $Lu$, respectively. Since $Lu_j^j(x) = f(x, u_j^j(x), \nabla u_j^j(x))$ by Lemma 3.1, it follows that $u(x)$ is a solution of (1) in $G_a$ of class $C^2(\overline{M})$, and hence of class $C^{2+\alpha}(\overline{M})$ by a standard regularity argument based on Schauder estimates. Since $w(x) \leqslant u_j^j(x) \leqslant v(x)$ for each $j = 1, 2, \ldots$, the function $u$ also satisfies $w(x) \leqslant u(x) \leqslant v(x)$ for all $x \in G_a$.

COROLLARY 3.4. *Let $\Omega$, $L$, and $\alpha$ be as in* (1) *and suppose that a positive number a has been selected so that $G_a \subset \Omega$. Suppose in addition to the other assumptions below* (1) *that $f(x, 0, 0) \leqslant 0$ for all $x \in G_a$. Then a necessary and sufficient condition for the existence of a nonnegative solution of* (1) *in $G_a$ is the existence of a nonnegative supersolution of* (1) *in $G_a$.*

This follows easily by the choice $w(x) \equiv 0$ in Theorem 3.3.

COROLLARY 3.5. *Let $\Omega$, $L$, $\alpha$, and a be as in Corollary* 3.4, *and suppose in addition to the assumptions* (i), (ii), *and* (iii) *(following* (1)*) that $f(x, u, p) \leqslant 0$ for all $x \in G_a$, $u \geqslant 0$, and $p \in R^n$. Then* (1) *has a positive solution in $G_a$ if there exists a nonnegative supersolution $v$ of* (1) *in $G_a \cup S_a$ with $v(x) > 0$ for at least one point $x \in S_a$.*

*Proof.* Let $u(x)$ be the nonnegative solution of (1) in $G_a$ implied by Corollary 3.4. Since $Lu = f(x, u, \nabla u) \leqslant 0$ in $G(a, a+i)$ for every positive integer $i$, and since $u = v \geqslant 0$ on $S_a \cup S_{a+i}$ with strict inequality at some point $x \in S_a$, the maximum principle [10] implies that $u(x) > 0$ throughout $G(a, a+i)$, and since $i$ is arbitrary, $u(x) > 0$ throughout $G_a$.

## 4. Bounds for Positive Solutions

In this section, (1) will be specialized to the form

$$\Delta u = f(x, u, \nabla u), \qquad x \in \Omega \tag{8}$$

under assumptions (ii) and (iii). If $f(x, u, p) \leqslant 0$ for all $x \in G_a$, $u \geqslant 0$, and $p \in R^n$, then the fundamental solution $w(x) = r^{2-n}$ $(n \geqslant 3)$ of Laplace's equation, where $r = |x|$, is a subsolution of (8) and is therefore a lower bound on the positive solution of (8) given by Theorem 3.3. Under the weaker assumption $f(x, 0, 0) \leqslant 0$ for all $x \in G_a$, the lower bound $w(x) = 0$ identically is given in Corollary 3.4. It will be shown below that $v(x) = r^q$ is a supersolution of (8) for suitable $q$ and suitable growth of $f(x, v(x), \nabla v(x))$. Then Theorem 3.3 ensures the existence of a solution $u(x)$ of (8) with these specific upper and lower bounds $v(x)$ and $w(x)$, respectively, throughout an exterior domain $G_a$ of $R^n$.

We shall require that the $x$-dependence of the Nagumo condition (iii) be more specific, as follows:

(iv) There exist nonnegative continuous functions

$$g: \Omega \to [0, \infty) \qquad \text{and} \qquad \phi: [0, \infty) \to [0, \infty)$$

such that

$$0 \leqslant -f(x, u, p) \leqslant g(x)\, \phi(|u|)\, (1 + |p|^2) \tag{9}$$

for all $x \in \Omega$, $u \in R^1$, and $p \in R^n$.

Corollary 4.1. *Under assumptions* (ii), (iii), *and* (iv), *the quasilinear Schrödinger equation* (8) *has a solution $u(x)$ in $G_a \subset R^n$ $(n \geqslant 3)$, for some $a > 0$, satisfying* $0 \leqslant u(x) \leqslant |x|^q$ *in $G_a$ if*

$$[\sup_{|x|=r} g(x)]\, \phi(r^q) \leqslant c r^{q-2}(1 + q^2 r^{2q-2})^{-1} \tag{10}$$

*for all $r \geqslant a$, where $q = 2 - n + \epsilon$, $c = -\epsilon q$, $0 < \epsilon < n - 2$.*

*Proof.* For $v(x) = r^q$, $r = |x|$, (9) and (10) give

$$r^{n-1}[\Delta v - f(x, v, \nabla v)] \leqslant \frac{d}{dr}\left(r^{n-1} \frac{d}{dr} r^q\right) + c r^{n+q-3} = [q(q + n - 2) + c]\, r^{q+n-3}$$
$$= 0.$$

Then $\Delta v \leqslant f(x, v, \nabla v)$ in $G_a$, and since $w(x) \equiv 0$ satisfies $\Delta w \geqslant f(x, w, \nabla w)$ and $w(x) \leqslant v(x)$ for all $x \in G_a$, the conclusion follows from Theorem 3.3.

In the special case that $\phi(t) = t^\gamma$, $\gamma > 0$ in (9), condition (10) is implied, for some $a > 0$, if

$$\limsup_{r\to\infty} r^\beta \hat{g}(r) < \epsilon(n - 2 - \epsilon), \tag{11}$$

where

$$\hat{g}(r) = \sup_{|x|=r} g(x), \qquad \beta = n - (n-2)\gamma + (\gamma - 1)\epsilon.$$

In fact, $\phi(r^q)\, r^{2-q}\hat{g}(r) = r^\beta \hat{g}(r) < c$ for all sufficiently large $r$ by (11), where

$$\beta = (\gamma - 1)\, q + 2 = n - (n-2)\gamma + (\gamma - 1)\epsilon,$$

which implies (10).

If $\gamma = 1$, i.e. (8) is linear in $u$, then $\beta = 2$, and if $\epsilon = \frac{1}{2}(n-2)$, then (11) reduces to the familiar Kneser criterion

$$\limsup_{r\to\infty} r^2 \hat{g}(r) < \frac{(n-2)^2}{4}.$$

COROLLARY 4.2. *Let $K$ and $\epsilon$ be positive numbers satisfying*

$$\limsup_{r\to\infty} r^{2+\epsilon}\hat{g}(r)\, \phi(K - r^{-\epsilon}) < \epsilon^2. \tag{12}$$

*Then under assumptions* (ii), (iii), *and* (iv) *there exists a positive number $a$ such that* (8) *has a solution $u(x)$ in $G_a \subset R^2$ satisfying*

$$0 \leqslant u(x) \leqslant K - |x|^{-\epsilon}. \tag{13}$$

*Proof.* Let $v(x) = K - r^{-\epsilon}$ where $r = |x|$, and use (9) and (12) to obtain

$$r[\Delta v - f(x, v, \nabla v)] \leqslant -\frac{d}{dr}\left(r \frac{d}{dr} r^{-\epsilon}\right) + \epsilon^2 r^{-\epsilon-1} = 0$$

for $|x| > a$ provided $a$ is sufficiently large. Then $\Delta v \leqslant f(x, v, \nabla v)$ in $G_a$ and the conclusion (13) again follows from Theorem 3.3.

Sharper bounds are obtained by the same procedure when (8) is a semilinear Schrödinger equation

$$\Delta u + F(x, u) = 0, \qquad x \in \Omega, \tag{14}$$

i.e. $F(x, u) = -f(x, u, p)$ is independent of $p$, under assumption (ii) and the nonnegativity assumption

(v) $0 \leqslant F(x, t)$ for all $x \in G_a$ and for all $t \geqslant 0$.

COROLLARY 4.3. *Under assumptions* (ii) *and* (v), *Equation* (14) *has a positive solution* $u(x)$ *in* $G_a \subset R^n$, $n \geqslant 3$, *for some* $a > 0$, *satisfying* $r^{2-n} \leqslant u(x) \leqslant r^{2-n+\epsilon}$, $x \in G_a$, *if*

$$\limsup_{r\to\infty} r^{n-\epsilon} \max_{|x|=r} F(x, |x|^{2-n+\epsilon}) < c, \tag{15}$$

*where* $c = \epsilon(n-2-\epsilon)$, $0 < \epsilon < n-2$.

In the case of the Emden–Fowler equation

$$\Delta u + g(x)\, u^\gamma = 0, \qquad \gamma > 0, \tag{16}$$

where $g(x) \geqslant 0$ in $\Omega$ and $g \in C^\alpha(\overline{M})$ for every bounded subdomain $M \subset \Omega \subset R^n$, $n \geqslant 3$, $0 < \alpha < 1$, the criterion (15) reduces to

$$\limsup_{r\to\infty} r^\beta \hat{g}(r) < \epsilon(n-2-\epsilon),$$

the same as the quasilinear condition (11).

COROLLARY 4.4. *Under assumptions* (ii) *and* (v), *there exists a positive number* $a$ *such that Equation* (14) *has a positive solution* $u(x)$ *in* $G_a \subset R^2$ *satisfying* $0 < K < u(x) \leqslant (\log r)^\epsilon$ *for any constant* $K > 0$ *if*

$$\limsup_{r\to\infty} r^2(\log r)^{2-\epsilon} \max_{|x|=r} F(x, (\log |x|^\epsilon) < c, \tag{17}$$

*where* $c = \epsilon(1-\epsilon)$, $0 < \epsilon < 1$.

In the Emden–Fowler case (16), the criterion (17) reduces to

$$\limsup_{r\to\infty} r^2(\log r)^b\, \hat{g}(r) < \epsilon(1-\epsilon),$$

where

$$b = 2 + (\gamma - 1)\,\epsilon, \qquad \hat{g}(r) = \sup_{|x|=r} g(x).$$

COROLLARY 4.5. *Let* $K_1$, $K_2$, *and* $\epsilon$ *be any positive numbers satisfying* $K_1 < K_2$ *and*

$$\limsup_{r\to\infty} r^{2+\epsilon} \max_{|x|=r} F(x, K_2) < \epsilon^2. \tag{18}$$

*Then under assumptions* (ii) *and* (v) *there exists a positive number* $a$ *such that* (14) *has a positive solution* $u(x)$ *in* $G_a \subset R^2$ *satisfying* $K_1 \leqslant u(x) \leqslant K_2 - r^{-\epsilon}$. *In particular*, (18) *is a sufficient condition for* (14) *to have a bounded positive solution in* $G_a$.

## 5. Nonoscillation Criteria

Equation (8) is called *nonoscillatory* in an unbounded domain $\Omega$ whenever (8) has a one-signed solution (as defined in Section 2) in $\Omega \cap G_r$ for some positive number $r$. Lemma 5.1 below reduces the nonoscillation problem for (8) to the corresponding problem for the ordinary differential inequality

$$\frac{d}{dr}\left(r^{n-1}\frac{d\rho}{dr}\right) + r^{n-1}\hat{g}(r)\phi(|\rho|)\left(1 + \left|\frac{d\rho}{dr}\right|^2\right) \leqslant 0. \tag{19}$$

Lemma 5.1. *If assumptions* (ii), (iii), *and* (iv) *hold, a sufficient condition for* (8) *to be nonoscillatory in an exterior domain* $\Omega \subset R^n$ *is the existence of a positive solution* $\rho \in C^{2+\alpha}[a, b]$ *of* (19) *for some numbers* $a > 0$ *and* $\alpha$, $0 < \alpha < 1$, *and for all* $b > a$, *where* $\hat{g}(r) = \sup_{|x|=r} g(x)$.

*Proof.* Condition (19) shows that the function $v$ defined in $G_a$ by $v(x) = \rho(r)$, $r = |x| \geqslant a$, satisfies

$$r^{n-1}[\Delta v - f(x, v, \nabla v)] \leqslant \frac{d}{dr}\left(r^{n-1}\frac{d\rho}{dr}\right) + r^{n-1}\hat{g}(r)\phi(\rho)\left(1 + \left|\frac{d\rho}{dr}\right|^2\right),$$

and hence $v(x)$ is a supersolution of (8) in $G_a$ by (19). Corollary 3.5 therefore implies that (8) has a positive solution in $G_a$.

In the Emden–Fowler case (16), (19) simplifies to

$$\frac{d}{dr}\left(r^{n-1}\frac{d\rho}{dr}\right) + r^{n-1}\hat{g}(r)\,[\rho(r)]^\gamma \leqslant 0, \qquad \gamma > 0. \tag{20}$$

The following assumptions on $g(x)$ are in effect for the theorems below:

(vi) $g(x) \geqslant 0$ for all $x \in \Omega$, $g \in C^\alpha(\overline{M})$ for every bounded subdomain $M \subset \Omega$, and $\hat{g} \in C^\alpha[a, b]$ for some $a > 0$, $0 < \alpha < 1$, and for all $b > a$.

Theorem 5.2. *Under assumption* (vi), *the sublinear Emden–Fowler equation* (16) (*i.e.* $0 < \gamma < 1$) *is nonoscillatory in an exterior domain in* $R^n$ *if*

$$\int_{r_0}^\infty r(\log r)^\gamma \hat{g}(r)\,dr < \infty, \qquad n = 2 \tag{21}$$

$$\int_{r_0}^\infty r\hat{g}(r)\,dr < \infty, \qquad n \geqslant 3 \tag{22}$$

*for some* $r_0 > 0$.

*Proof.* For $n = 2$, Liouville's substitution $r = e^s$, $h(s) = \rho(e^s)$ transforms (20) (in the case of equality) into

$$h''(s) + e^{2s}\hat{g}(e^s)\,[h(s)]^\gamma = 0, \tag{23}$$

which has a positive solution $h_0(s)$ for sufficiently large $s$ if (21) is satisfied by Belohorec's well known theorem [3, 12]. Then (21) is sufficient for (16) to have a positive solution $\rho_0(r)$ in $[a, \infty)$ for some positive number $a$. Since $\hat{g} \in C^\alpha[a, b]$ for all $b > a$, standard regularity theory [6] shows that $\rho_0 \in C^{2+\alpha}[a, b]$ for all $b > a$. Theorem 5.2 in the case $n = 2$ is therefore a consequence of Lemma 5.1. For $n \geqslant 3$ the proof is virtually the same after transformation of (20) to

$$h''(s) + s^{-3-\gamma}[\beta(s)]^{2n-2}\,\hat{g}[\beta(s)]\,[h(s)]^\gamma = 0 \tag{24}$$

by the substitution $r = \beta(s) = (\nu s)^\nu$, $h(s) = s\rho[\beta(s)]$, where $\nu = 1/(n-2)$.

Theorem 5.3. *Under assumption* (vi), *the superlinear Emden–Fowler equation* (16) (*i.e.* $\gamma > 1$) *is nonoscillatory in an exterior domain in* $R^n$ *if*

$$\int_{r_0}^\infty r \log r\hat{g}(r)\,dr < \infty, \qquad n = 2 \tag{25}$$

$$\int_{r_0}^\infty r^\sigma \hat{g}(r)\,dr < \infty, \qquad n \geqslant 3 \tag{26}$$

*for some* $r_0 > 0$, *where* $\sigma = n - 1 - \gamma(n-2)$.

The proof is the same as that of Theorem 5.2, except Atkinson's criterion [2] for (23) or (24) to have an eventually positive solution $h_0(s)$ is applied instead of Belohorec's criterion.

Theorem 5.4. *Under assumption* (vi), *condition* (25) ((26), *respectively*) *is sufficient for* (16) *to have a bounded positive solution in some exterior domain* $G_a \subset R^2$ ($G_a \subset R^n$, $n \geqslant 3$, *respectively*).

*Proof.* The transformed equations (23) and (24) have the canonical form $h''(s) + Q(s)\,h^\gamma(s) = 0$ for which Nehari's criterion [8, p. 103]

$$\int_{s_0}^\infty sQ(s)\,ds < \infty \tag{27}$$

for the existence of a bounded eventually positive solution $h(s)$ is applicable, and (27) is equivalent to (25), (26) for $n = 2$, $n \geqslant 3$, respectively.

The corollary below shows that (22) in fact *characterizes* nonoscillatory sublinear equations (16) provided that $g(x)$ does not fluctuate too wildly on the sphere $S_r \subset R^n$, $n \geqslant 3$. For this result, $g_M(r)$ denotes the spherical mean of $g(x)$ over $S_r$:

$$g_M(r) = \frac{1}{\omega(S_1)} \int_{S_1} g(r, \theta)\,d\omega,$$

where $r$, $\theta$ denote hyperspherical coordinates and $\omega$ denotes the measure on the unit sphere $S_1$. Our extra hypothesis is

$$\liminf_{r\to\infty} g_M(r)/\hat{g}(r) > 0. \tag{28}$$

COROLLARY 5.5. *If $g(x)$ satisfies* (28) *and the hypotheses of Theorem* 5.2, *then* (22) *is a necessary and sufficient condition for the sublinear Emden–Fowler equation* (16) *to be nonoscillatory in an exterior domain $\Omega \subset R^n$, $n \geqslant 3$.*

*Proof.* The sufficiency of (22) is contained in Theorem 5.2. If (22) fails, then by (28) there exist positive numbers $\epsilon$ and $r_0$ such that

$$\int_{r_0}^{\infty} r g_M(r)\,dr \geqslant \epsilon \int_{r_0}^{\infty} r\hat{g}(r)\,dr = +\infty$$

and (16) is oscillatory by a theorem of Kitamura and Kusano [5].

Similarly one can prove the following on the basis of [9, Theorems 11 and 12].

COROLLARY 5.6. *If $g(x)$ satisfies* (28) *and the hypotheses of Theorem* 5.3, *then* (25) [(26), *respectively*] *is a necessary and sufficient condition for the superlinear Emden–Fowler equation* (16) *to be nonoscillatory in an exterior domain in $R^2$ [$R^n$ ($n \geqslant 3$), respectively].*

Corollaries 5.5 and 5.6 are the only characterizations of oscillatory partial differential equations discovered to date, and cover all the strictly nonlinear cases of (16) except $n = 2$, $0 < \gamma < 1$. Further exploration of this topic would require additional study of the *ordinary* differential inequality (19) and would carry us outside the theme of this paper.

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